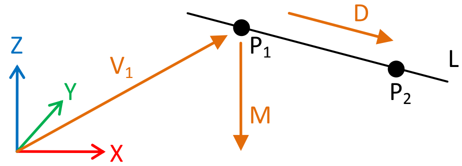
Plücker Line

A Plücker representation for a 3-dimensional line is:



$$\{D_x, D_y, D_z \quad ; \quad M_x, M_y, M_z\}$$

Where (D_x, D_y, D_z) is the direction vector $\overline{D}$ and (M_x, M_y, M_z) is the moment vector $\overline{M}$ of the line. This representation of the line is contains the minimal number of parameters to describe a 3-dimensional line and is capable of retaining direction information even if the line is infinitely far away from the coordinate system origin.

The direction vector can be calculated as the unitized delta between two points.

$$D = \frac{P_2 - P_1}{|P_2 - P_1|}$$

Where:

$$|P_2 - P_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 \cdot (z_2 - z_1)^2}$$

The directional components are therefore:

$$D_x = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 \cdot (z_2 - z_1)^2}}$$

$$D_y = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 \cdot (z_2 - z_1)^2}}$$

$$D_z = \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 \cdot (z_2 - z_1)^2}}$$

The moment vector is calculated by taking the cross product of any point on the line and the direction vector. The first point is used for simplicity, preventing the need to define any additional points. The point's coordinates define the vector from the origin to the point's location, so the point coordinates are used for the cross product.

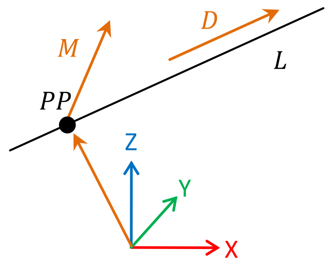
$$M = P_1 \times D$$

Expanded out, the moment vector is calculated as:

$$M_x = y_1 D_z - z_1 D_y$$
$$M_y = z_1 D_x - x_1 D_z$$
$$M_z = x_1 D_y - y_1 D_x$$

Principal Point

The principal point (PP) of the Plücker line is the point at which the line passes closest to the origin.



This value can be calculated by taking the cross product of the unit direction vector D and the moment vector M.

$$PP = D \times M$$

To unitize the direction vector the whole line must be divided by the magnitude of the direction vector m.

$$m = \sqrt{D_x^2 + D_y^2 + D_z^2}$$

The line's components are then updated by dividing all the components of the line by m.

$$\widehat{D}_x = \frac{D_x}{m} \qquad M_x = \frac{M_x}{m}$$
$$\widehat{D}_x = \frac{D_y}{m} \qquad M_y = \frac{M_y}{m}$$
$$\widehat{D}_x = \frac{D_z}{m} \qquad M_z = \frac{M_z}{m}$$

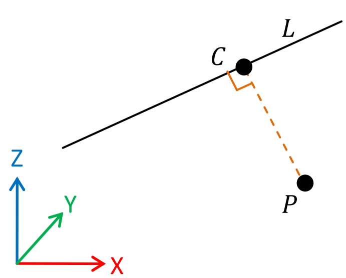
This is the standard representation of the Plücker line. The direction vector is stored as a unit vector, and the magnitude of the moment vector is equal to the magnitude of the vector from the origin to the principal point of the line. If the line is stored in standard representation, the unitization step is not required.

To calculate the principal point, take the cross product of the unit direction vector $\widehat{D}$ and moment vector M.

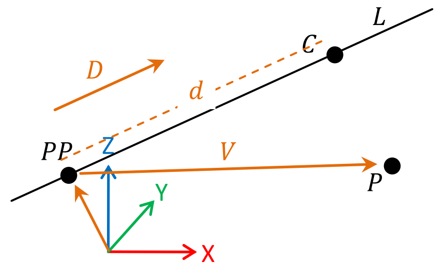
$$PP_x = \widehat{D}_y \cdot M_z - \widehat{D}_z \cdot M_y$$
$$PP_y = \widehat{D}_z \cdot M_x - \widehat{D}_x \cdot M_z$$
$$PP_z = \widehat{D}_x \cdot M_y - \widehat{D}_y \cdot M_x$$

Point of Closest Approach

To find the point of closest approach to the line (C) for a point in space (P) as shown below:



The principal point (PP) can be used to define a starting point on the line. The distance d along the line from the point PP to C must be calculated.



The distance d could be calculated by projecting the vector V onto the direction vector of the line, but since the origin is normal to the prinicipal point in the plane of $\overline{PP - C - P}$ the value d can be calculated using the vector from the origin to the point.

$$d = V \circ \hat{D} = P \circ \hat{D}$$

The principal point can be calculated as:

$$PP_x = \hat{D}_y \cdot M_z - \hat{D}_z \cdot M_y$$
$$PP_y = \hat{D}_z \cdot M_x - \hat{D}_x \cdot M_z$$
$$PP_z = \hat{D}_x \cdot M_y - \hat{D}_y \cdot M_x$$

The vector V can then be projected onto the line by taking the dot product of the vector and the unit direction vector of the line $\hat{D}$.

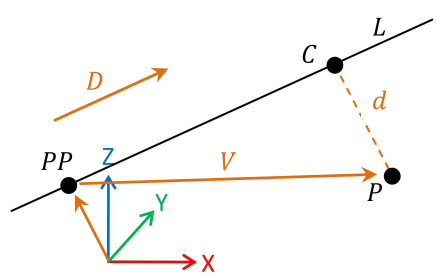
$$d = P_x \cdot \hat{D}_x + P_y \cdot \hat{D}_y + P_z \cdot \hat{D}_z$$

The closest point can then be calculated by adding the scaled direction vector to the principal point

$$C_x = PP_x + \hat{D}_x \cdot d$$
$$C_y = PP_y + \hat{D}_y \cdot d$$
$$C_z = PP_z + \hat{D}_z \cdot d$$

Distance from Point to Line

The distance from a point to the line can be calculated using the principal point (PP) and some simple trigonometry.



The principal point is defined by:

$$PP_x = \widehat{D}_y \cdot M_z - \widehat{D}_z \cdot M_y$$
$$PP_y = \widehat{D}_z \cdot M_x - \widehat{D}_x \cdot M_z$$
$$PP_z = \widehat{D}_x \cdot M_y - \widehat{D}_y \cdot M_x$$

The vector between the principal point and the point of interest is calculated by:

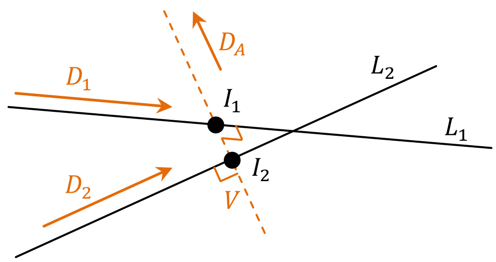
$$V_x = P_x - PP_x$$
$$V_y = P_y - PP_y$$
$$V_z = P_z - PP_z$$

Taking the dot product of the principal point vector and the vector V gives the distance from the point to the line.

$$d = (P_x - PP_x) \cdot PP_x + (P_y - PP_y) \cdot PP_y + (P_z - PP_z) \cdot PP_z$$

Intersecting Two Lines

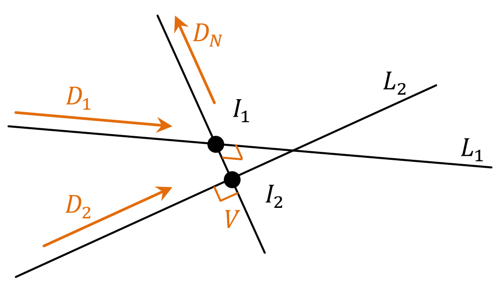
[TBD]



[TBD]

Calculate Intersecting Normal Line

[TBD]



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