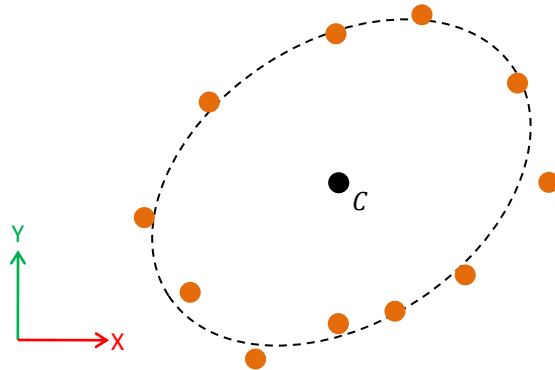


Eigensolution for Best Fit Ellipse

A best-fit ellipse can be found for a collection of points using the Eigensolution of the covariance matrix of the points. Take the series of points in the example below:



First the mean of the points (i.e. centroid) is calculated.

$$x_c = \frac{\sum x_i}{n}$$
$$y_c = \frac{\sum y_i}{n}$$

The points are then normalized by moving them to the origin of the system (i.e. subtracting the centroid from each point).

$$x'_i = x_i - x_c$$
$$y'_i = y_i - y_c$$

The covariance is then calculated as:

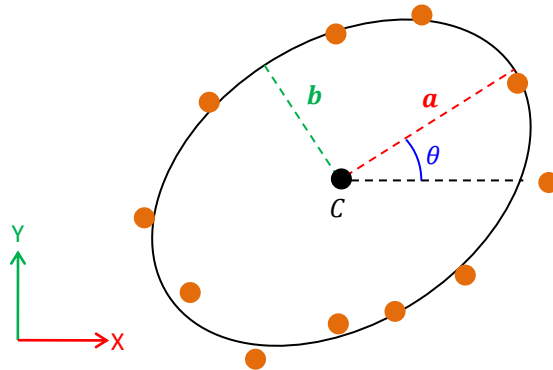
$$A = \begin{bmatrix} \sum x_i'^2 & \sum x'_i y'_i \\ \sum x'_i y'_i & \sum y_i'^2 \end{bmatrix}$$

The eigenvectors and values of this symmetric matrix are then calculated. *Note: The eigensolution of a 2x2 matrix is detailed in a separate section [here](#).*

$$[V_1, V_2, \lambda_1, \lambda_2] = \text{SolveEigen2x2}(A)$$

The eigenvectors (V_1, V_2) of the matrix A represent the vector directions of the major and minor axes of the ellipse. The associated eigenvalues (λ_1, λ_2) are scalar multiples of the major and minor axis lengths. The eigenvectors must first be unitized.

$$\hat{V}_1 = \frac{V_1}{|V_1|} \quad \text{and} \quad \hat{V}_2 = \frac{V_2}{|V_2|}$$



The major axis (\mathbf{a}) will be defined by the largest eigenvalue and its associated eigenvector. The minor axis (\mathbf{b}) will be defined by the other.

$$\begin{aligned}
 &\text{if}(\lambda_1 > \lambda_2) \\
 &\quad \hat{\mathbf{v}}_a = \hat{\mathbf{v}}_1 \\
 &\quad \hat{\mathbf{v}}_b = \hat{\mathbf{v}}_2 \\
 &\quad \lambda_a = \lambda_1 \\
 &\quad \lambda_b = \lambda_2 \\
 &\text{else} \\
 &\quad \hat{\mathbf{v}}_a = \hat{\mathbf{v}}_2 \\
 &\quad \hat{\mathbf{v}}_b = \hat{\mathbf{v}}_1 \\
 &\quad \lambda_a = \lambda_2 \\
 &\quad \lambda_b = \lambda_1
 \end{aligned}$$

The eigenvalues are a scaled representation of the major and minor axes, not the lengths themselves. To calculate the semi-major and semi-minor axis lengths, we need to normalize by the number of points used.

$$\begin{aligned}
 a &= |\mathbf{a}| = 4 \cdot \sqrt{\lambda_a/N} \\
 b &= |\mathbf{b}| = 4 \cdot \sqrt{\lambda_b/N}
 \end{aligned}$$

The final major and minor axis vectors are calculated by scaling the unitized eigenvectors by the axis lengths.

$$\begin{aligned}
 \mathbf{a} &= \hat{\mathbf{v}}_a \cdot a \\
 \mathbf{b} &= \hat{\mathbf{v}}_b \cdot b
 \end{aligned}$$

The ellipse can also be described parametrically as:

$$\begin{aligned}
 x &= x_c + a \cdot \cos(2\pi t) \cdot \cos(\theta) - b \cdot \sin(2\pi t) \cdot \sin(\theta) \\
 y &= y_c + a \cdot \cos(2\pi t) \cdot \sin(\theta) + b \cdot \sin(2\pi t) \cdot \cos(\theta)
 \end{aligned}$$

Where t is the independent parameter and θ is the rotational angle of the major axis from the x-axis.