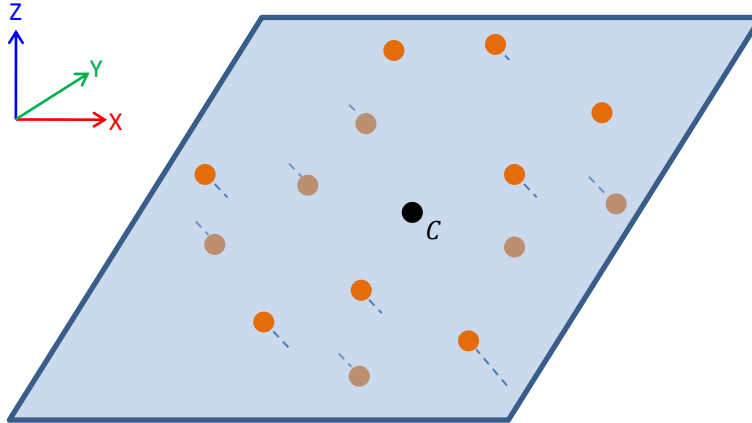


## Eigensolution for Best-Fit 3D Ellipse / Best-Fit Plane

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A best-fit 3D ellipse (or plane) can be found for a collection of points using the Eigensolution of the covariance matrix of the points. Take the series of points in the example below:



First the mean of the points (i.e. centroid) is calculated.

$$C_x = \frac{\sum x_i}{n}$$
$$C_y = \frac{\sum y_i}{n}$$
$$C_z = \frac{\sum z_i}{n}$$

The points are then normalized by moving them to the origin of the system (i.e. subtracting the centroid from each point).

$$x'_i = x_i - C_x$$
$$y'_i = y_i - C_y$$
$$z'_i = z_i - C_z$$

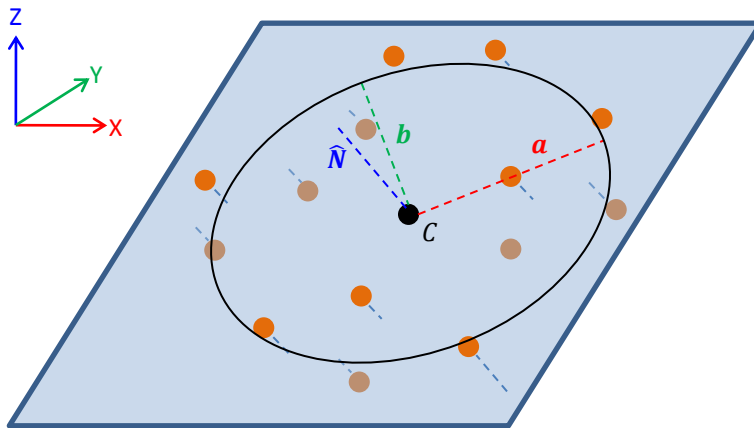
The covariance is then calculated as:

$$A = \begin{bmatrix} \sum x_i'^2 & \sum x'_i y'_i & \sum x'_i z'_i \\ \sum x'_i y'_i & \sum y_i'^2 & \sum y'_i z'_i \\ \sum x'_i z'_i & \sum y'_i z'_i & \sum z_i'^2 \end{bmatrix}$$

The eigenvectors and values of this symmetric matrix are then calculated. *Note: The eigensolution of a 3x3 matrix is detailed in a separate section [here](#).*

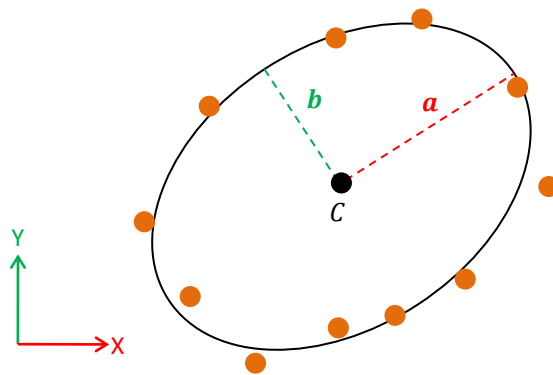
$$\begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \text{SolveEigen3x3}(A)$$

The eigenvectors ( $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ ) of the matrix  $A$  represent orthogonal axes of the best-fit ellipse. The eigenvector associated with the smallest eigenvalue represents the normal direction ( $\hat{N}$ ) of the plane. The eigenvector associated with the largest eigenvalue represents a scalar multiple of the major axis ( $a$ ) of an ellipse on the plane and the mid value represents a scalar multiple of the minor axis ( $b$ ) of the ellipse.



the vector directions of the major and minor axes of the ellipse. The associated eigenvalues ( $\lambda_1, \lambda_2$ ) are scalar multiples of the major and minor axis lengths. The eigenvectors must first be unitized.

$$\hat{V}_1 = \frac{V_1}{|V_1|} \text{ and } \hat{V}_2 = \frac{V_2}{|V_2|}$$



The major axis ( $\mathbf{a}$ ) will be defined by the largest eigenvalue and its associated eigenvector. The minor axis ( $\mathbf{b}$ ) will be defined by the other.

$$\begin{aligned} &\text{if}(\lambda_1 > \lambda_2) \\ &\quad \hat{V}_a = \hat{V}_1 \\ &\quad \hat{V}_b = \hat{V}_2 \\ &\quad \lambda_a = \lambda_1 \\ &\quad \lambda_b = \lambda_2 \\ &\text{else} \\ &\quad \hat{V}_a = \hat{V}_2 \\ &\quad \hat{V}_b = \hat{V}_1 \\ &\quad \lambda_a = \lambda_2 \\ &\quad \lambda_b = \lambda_1 \end{aligned}$$

The eigenvalues are a scaled representation of the major and minor axes, not the lengths themselves. To calculate the semi-major and semi-minor axis lengths, we need to normalize by the number of points used.

$$|a| = 4 \cdot \sqrt{\lambda_a / N}$$

$$|b| = 4 \cdot \sqrt{\lambda_b / N}$$

The final major and minor axis vectors are calculated by scaling the unitized eigenvectors by the axis lengths.

$$\begin{aligned}\mathbf{a} &= \hat{\mathbf{V}}_a \cdot |a| \\ \mathbf{b} &= \hat{\mathbf{V}}_b \cdot |b|\end{aligned}$$