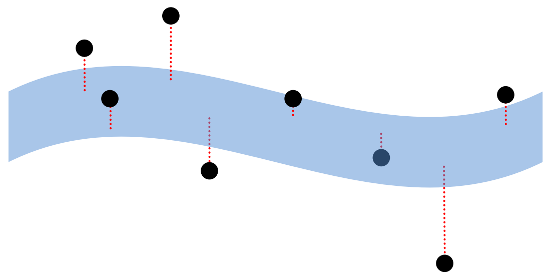
Bipolynomial Regression

Bi-polynomial regression calculates a best fit polynomial surface through a group of n or more data points (where n represents the total number of variables in the polynomial surface equation). The surface is calculated by minimizing the residuals (or errors) between the surface and the original points using least squares minimization. The figure below represents a 2[rd] order surface.



The least squares minimization equation is:

$$\varepsilon^2 = \sum_{i=1}^{n} (f(x_i, y_i) - z_i)^2$$

Where z_i are the observed values and $f(x_i, y_i)$ is the y-value of the surface at x_i, y_i. The equation of the surface is:

$$z = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

Plugging this value in to the regression equation gives

$$\varepsilon = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i)^2$$

To find the minimum residual error, the derivative of the residuals equation must be zero, which means all of the partial derivatives with respect to each coefficient must be equal to zero.

$$\frac{\partial\varepsilon}{\partial A} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) \cdot x_i^2 = 0$$

$$\frac{\partial\varepsilon}{\partial B} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) \cdot x_iy_i = 0$$

$$\frac{\partial\varepsilon}{\partial C} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) \cdot y_i^2 = 0$$

$$\frac{\partial\varepsilon}{\partial D} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) \cdot x_i = 0$$

$$\frac{\partial\varepsilon}{\partial E} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) \cdot y_i = 0$$

$$\frac{\partial\varepsilon}{\partial F} = \sum_{i=1}^{n} (Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F - z_i) = 0$$

These equations can be expressed in matrix form:

$$\begin{bmatrix} \sum x_i^4 & \sum x_i^3 y_i & \sum x_i^2 y_i^2 & \sum x_i^3 & \sum x_i^2 y_i & \sum x_i^2 \\ \sum x_i^3 y_i & \sum x_i^2 y_i^2 & \sum x_i y_i^3 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i y_i \\ \sum x_i^2 y_i^2 & \sum x_i y_i^3 & \sum y_i^4 & \sum x_i y_i^2 & \sum y_i^3 & \sum y_i^2 \\ \sum x_i^3 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i^2 y_i & \sum x_i y_i^2 & \sum y_i^3 & \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i^2 & \sum x_i y_i & \sum y_i^2 & \sum x_i & \sum y_i & n \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} x_i^2 z_i \\ x_i y_i z_i \\ y_i^2 z_i \\ x_i z_i \\ y_i z_i \\ z_i \end{bmatrix}$$

Solving for A - F

$$\begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} \sum x_i^4 & \sum x_i^3 y_i & \sum x_i^2 y_i^2 & \sum x_i^3 & \sum x_i^2 y_i & \sum x_i^2 \\ \sum x_i^3 y_i & \sum x_i^2 y_i^2 & \sum x_i y_i^3 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i y_i \\ \sum x_i^2 y_i^2 & \sum x_i y_i^3 & \sum y_i^4 & \sum x_i y_i^2 & \sum y_i^3 & \sum y_i^2 \\ \sum x_i^3 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i^2 y_i & \sum x_i y_i^2 & \sum y_i^3 & \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i^2 & \sum x_i y_i & \sum y_i^2 & \sum x_i & \sum y_i & n \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_i^2 z_i \\ x_i y_i z_i \\ y_i^2 z_i \\ x_i z_i \\ y_i z_i \\ z_i \end{bmatrix}$$