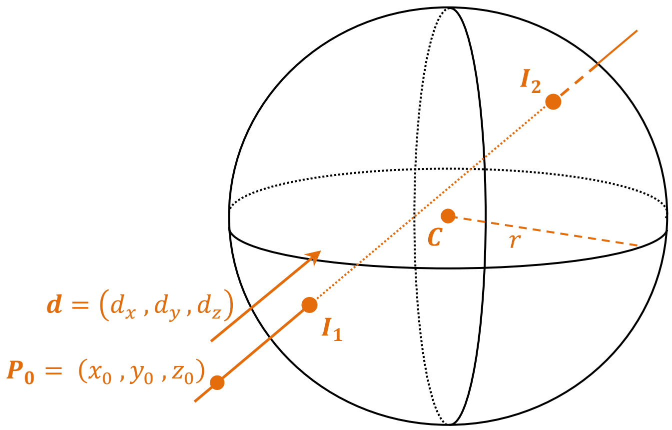
Intersection of Parametric Line and Sphere

A 3D line and a sphere can intersect at 0, 1, or 2 points.



Using a parametric line in the form of:

$$x = x_0 + d_x \cdot t$$
$$y = y_0 + d_y \cdot t$$
$$z = z_0 + d_z \cdot t$$

and a sphere in the form of:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

The intersection points I_1 and I_2 can be calculated by combining the two equations.

$$(x_0 + d_x \cdot t - x_c)^2 + (y_0 + d_y \cdot t - y_c)^2 + (z_0 + d_z \cdot t - z_c)^2 = r^2$$

Expanding this equation out:

$$x_0^2 + 2x_0 d_x t - 2x_0 x_c - 2d_x x_c t + d_x^2 t^2 + x_c^2 +$$
$$y_0^2 + 2y_0 d_y t - 2y_0 y_c - 2d_y y_c t + d_y^2 t^2 + y_c^2 +$$
$$z_0^2 + 2z_0 d_z t - 2z_0 z_c - 2d_z z_c t + d_z^2 t^2 + z_c^2 = r^2$$

And grouping by t:

$$[d_x^2 + d_y^2 + d_z^2]\, t^2 +$$
$$[2d_x(x_0 - x_c) + 2d_y(y_0 - y_c) + 2d_z(z_0 - z_c)]\, t$$
$$[x_0^2 + x_c^2 - 2x_0 x_c + y_0^2 + y_c^2 - 2y_0 y_c + z_0^2 + z_c^2 - 2z_0 z_c] = r^2$$

A quadradic equation can be created in the form of:

$$at^2 + bt + c = 0$$

Where:

$$
\begin{aligned}
a &= d_x^2 + d_y^2 + d_z^2 \\
b &= 2\left[d_x(x_0 - x_c) + d_y(y_0 - y_c) + d_z(z_0 - z_c)\right] \\
c &= x_0^2 + x_c^2 - 2x_0x_c + y_0^2 + y_c^2 - 2y_0y_c + z_0^2 + z_c^2 - 2z_0z_c - r^2
\end{aligned}
$$

c can be simplified as:

$$c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2$$

The values of t are then calculated using the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value(s) of t represent the distance along the parametric line of the intersection points. If both roots of t are real (i.e. the discriminant is positive), the line intersects the sphere and the two values of t can be used to solve for points I_1 and I_2. If the roots are the same (i.e. the discriminant is 0) the line is tangential to the sphere and only intersects at the one point. If both roots are complex (i.e. the discriminant is negative) the line does not intersect the sphere.

$$
\begin{aligned}
I_{xi} &= x_0 + d_x \cdot t_i \\
I_{yi} &= y_0 + d_y \cdot t_i \\
I_{zi} &= z_0 + d_z \cdot t_i
\end{aligned}
$$

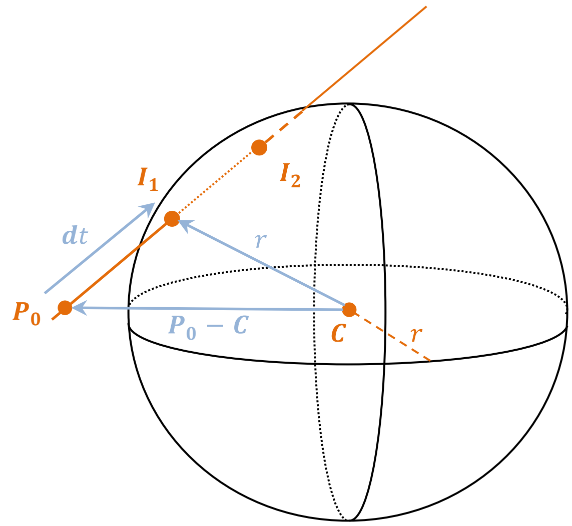
Note: More intuition about the process can be gained by converting a, b, and c, to vector form and simplifying.

$$
\begin{aligned}
a &= \boldsymbol{d} \circ \boldsymbol{d} \\
b &= 2\boldsymbol{d} \circ (\boldsymbol{P_0} - \boldsymbol{C}) \\
c &= (\boldsymbol{P_0} - \boldsymbol{C}) \circ (\boldsymbol{P_0} - \boldsymbol{C}) - r^2
\end{aligned}
$$

Which breaks down to the equation:

$$\left(\boldsymbol{d}t + (\boldsymbol{P_0} - \boldsymbol{C})\right)^2 = r^2$$

Looking at this visually, the process is apparent. We're calculating the location of I as summation of vectors that meets the radius condition.



The vector $P_0 - C$ gives the distance from the center of the sphere to the start point of the line and the vector $\boldsymbol{d}t$ moves from the start of the line to the intersection point. A value of t must then be defined that makes the magnitude of this vector sum equal to r.